1. Numbers and Limits.

1.1. Inequalities.

① Law of Trichotomy : $\forall a,\ a<0$ or $a=0$ or $a>0$

② The Triangle Inequality : $|a+b| \leq |a|+|b|$

③ The A-G Inequality : $\forall a,b>0,\ \frac{a+b}{2} \geq \sqrt{ab}$

(with equality only for $a=b$)

↳ Genalized Version.

$\forall a_n > 0$, for $n \in \mathbb{N}$.

$$\frac{a_1+a_2+\cdots+a_n}{n} \geq (a_1 a_2 \cdots a_n)^{\frac{1}{n}}$$

1.2. Numbers and the Least Upper Bound Theorem.

Def. Upper Bound of Set S

: number k s.t. $x \leq k$ for $\forall x \in S$.

Set S is bounded above $\Leftrightarrow$ There is Upper Bound of S.

Least Upper Bound of S : number L s.t. $L \leq k$ for k is UB of S

Lower Bound of set S / Set is bounded below / Greatest Lower Bound : Analogous.

Thm 1.2. Least Upper Bound Theorem.

: Every Set S that is Bounded above has a Least Upper Bound.

Thm 1.3. Greatest Lower Bound Theorem

: Analogous.

1.3. Sequences and Their Limits. (+ Series).

Def. Sequence : A list of numbers.

Sequence $\{a_n\}$ converges to the number a : $\lim_{n\to\infty} a_n = a$

$\Leftrightarrow$ $\forall \varepsilon > 0,\ \exists N \in \mathbb{N}$ s.t. if $n > N$, then $|a_n - a| < \varepsilon$.

- the number a is called the limit of $\{a_n\}$.
- Sequence converges to number a $\Rightarrow$ Sequence is convergent.

Sequence $\{a_n\}$ diverges (is divergent) : Sequence has no limit.

1.3. Sequences and Their Limits. (+ Series) 계속.

Thm 1.6. Suppose $\{a_n\}$ and $\{b_n\}$ are convergent, and $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$. then,

(a) $\lim_{n\to\infty} (a_n + b_n) = a+b$.

(b) $\lim_{n\to\infty} (a_n b_n) = ab$.

(c) if $a \neq 0$, then ① for n large enough, $a_n \neq 0$

② $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$

Thm 1.7. Suppose for $n > N$, $a_n \le b_n \le c_n$.

and that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = a$.

Then, $\lim_{n\to\infty} b_n = a$.

Def. sequence $\{a_n\}$ is increasing : for $n \in \mathbb{N}$, $a_n \le a_{n+1}$.

sequence $\{a_n\}$ is decreasing : for $n \in \mathbb{N}$, $a_n \ge a_{n+1}$.

sequence $\{a_n\}$ is monotonic : either $\{a_n\}$ is increasing or decreasing.

Def. sequence $\{a_n\}$ is bounded : $\exists B > 0$ s.t. $|a_n| \le B$ for $n \in \mathbb{N}$.

- The number B is called Bound.

sequence $\{a_n\}$ is bounded above by K

: $\exists K$ s.t. $a_n < K$ for $n \in \mathbb{N}$.

sequence $\{a_n\}$ is bounded below by K

: $\exists K$ s.t. $a_n > K$ for $n \in \mathbb{N}$.

Thm. 1.8. Every convergent sequence is bounded.

Thm 1.10. The monotone convergence theorem.

: A bounded monotone sequence converges.

Def. Geometric Sequence: Sequence of numbers that follow the pattern of multiplying by a fixed number to get the next number.

Thm. 1.11. the sequence $\{r^n\}$

(a) Converges to 0 if $|r|<1$

(b) Converges to 1 if $r=1$

(c) diverges for $r\leq -1$, $r>1$

Def. The sequence $\{a_n\}$ exist, then we define

⇒ ① **Partial Sums**: $S_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$

② **Series**: $\lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} a_n$

if ② exist, then we say the series converges.

otherwise, it diverges.

Thm. 1.12. if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Thm. 1.13. Geometric Series. if $|\alpha|<1$, $S_n = 1+\alpha+\alpha^2+\cdots+\alpha^n$ converges,

and $\lim_{n\to\infty} S_n = \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$.

if $|\alpha|>1$, it diverges.

Thm. 1.14. Comparison theorem: Suppose that for all n, $0\leq a_n \leq b_n$

if $\sum_{n=1}^{\infty} b_n$ conv., $\sum_{n=1}^{\infty} a_n$ conv.

㊀ 대우 : for all n, $0\leq a_n \leq b_n$, if $\sum_{n=1}^{\infty} a_n$ div., $\sum_{n=1}^{\infty} b_n$ div.

Thm. 1.15. Limit Comparison Theorem. Suppose that for all n, $a_n>0$ and $b_n>0$

if $\lim_{n\to\infty} \frac{a_n}{b_n}$ exist and is a positive number, then

$\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges

㊀ 대우: ~, $\sum_{n=1}^{\infty} a_n$ diverges if and only if $\sum_{n=1}^{\infty} b_n$ diverges

Thm 1.16. If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

↳ later, we say $\sum_{n=1}^{\infty} a_n$ absolutely converges

1.3. Sequences and Their Limits. (+ Series) 계속.

Thm. 1.17. Alternating Series Theorem.

If a_n is positive, decreasing, $\lim_{n\to\infty} a_n = 0$,

then, $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

Def 1.7. We say $\sum_{n=1}^{\infty} a_i$ converges conditionally

If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} |a_n|$ diverges.

We say $\sum_{n=1}^{\infty} a_i$ converges absolutely If both converges.

Thm 1.18. Ratio test. Suppose that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, then

(a) if $L<1$, $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(b) if $L>1$, $\sum_{n=1}^{\infty} a_n$ diverges.

if $L=1$, nothing is conclusive.

Def. Cauchy's criterion.

A sequence $\{a_n\}$ is called a Cauchy sequence

if $\forall \varepsilon>0, \exists N \in \mathbb{N}$ s.t. if $n, m > N$,

then $|a_n - a_m| < \varepsilon$.

Thm. 1.20. Every Cauchy sequence is convergent.

1.4. The Number e.

Def. Number e is the limit of Sequence e_n, where $e_n = \left(1+\frac{1}{n}\right)^n$

※ Let's show that e_n converges.

① Monotonicity of e_n : Let's think $n+1$ numbers, which are $\underbrace{\left(1+\frac{1}{n}\right), \left(1+\frac{1}{n}\right), \cdots, \left(1+\frac{1}{n}\right)}_{n \text{ times.}}, 1$

n is positive, so each terms are positive. Let's use A-G Inequality

$$\left.\begin{aligned} AM &= \frac{n\cdot\left(1+\frac{1}{n}\right) + 1}{n+1} = \frac{n+1+1}{n+1} = 1+\frac{1}{n+1} \\ GM &= \left\{\left(1+\frac{1}{n}\right)^n\right\}^{\frac{1}{n+1}} \end{aligned}\right\} \Rightarrow AM \geq GM\text{이므로}\quad AM^{n+1} \geq GM^{n+1}$$

So, $\left(1+\frac{1}{n+1}\right)^{n+1} \geq \left(1+\frac{1}{n}\right)^n$, $e_{n+1} \geq e_n$.

② Boundeness of e_n : let define $f_n = \left(1+\frac{1}{n}\right)^{n+1}$.

We know that $\left(1+\frac{1}{n}\right) > 1$, So $e_n < f_n$.

※ Let's think $n+1$ numbers, which are $\underbrace{\left(1-\frac{1}{n}\right), \cdots, \left(1-\frac{1}{n}\right)}_{n \text{ times.}}, 1$

let's use AM-GM inequality.

$$\left.\begin{aligned} AM &= \frac{n\left(1-\frac{1}{n}\right)+1}{n+1} = \frac{n}{n+1} \\ GM &= \left\{\left(1-\frac{1}{n}\right)^n\right\}^{\frac{1}{n+1}} \end{aligned}\right\} \Rightarrow AM^{n+1} \geq GM^{n+1}.$$

$$\text{so, } \left(\frac{n}{n+1}\right)^{n+1} \geq \left(\frac{n-1}{n}\right)^n \Leftrightarrow \left(\frac{n}{n-1}\right)^n \geq \left(\frac{n+1}{n}\right)^{n+1}$$

$$\Leftrightarrow \left(1+\frac{n}{n-1}\right)^n \geq \left(1+\frac{1}{n}\right)^{n+1}$$

which means $f_n \leq f_{n-1}$.

By the box above, we know $e_n < f_n < f_{n-1} < \cdots < f_1$,

$f_1 = \left(1+\frac{1}{1}\right)^2 = 4$, So $e_n < 4$.

By ① and ②, and By Monotone Convergence Theorem, e_n Converges, and the limit is number e.

Calculus

Series Convergence Tests

Test	Series	Converges	Diverges	Remarks
For Divergence (TFD)	$\sum_{n=1}^{\infty} a_n$	CANNOT show convergence	$\lim_{n\to\infty} a_n \neq 0$	always check first!
Geometric	$\sum_{n=1}^{\infty} ar^{n-1}$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	$\text{sum} = \dfrac{\text{first term}}{1-r}$
Telescoping	$\sum_{n=1}^{\infty}\left(b_n - b_{n+k}\right)$	$\lim_{n\to\infty} b_{n+k} = L$ L has to be finite	$\lim_{n\to\infty} b_{n+k}$ D.N.E. or inf	write out serval terms then cancel stuff to find paritial sum
P-Series	$\sum_{n=1}^{\infty} \dfrac{1}{n^p}$	$p > 1$	$p \leq 1$	famous sum $\sum_{n=1}^{\infty} \dfrac{1}{n^2} = \dfrac{\pi^2}{6}$
Integral	$\sum_{n=1}^{\infty} a_n$ $a_n = f(n) \geq 0$	$\int_1^{\infty} f(x)\, dx$ Converges	$\int_1^{\infty} f(x)\, dx$ Diverges	$f(x)$ has to be positive, continuous & decreasing for $x \geq 1$
Direct Comparison (DCT)	$\sum_{n=1}^{\infty} a_n$ $a_n > 0$	$\sum_{n=1}^{\infty} a_n \leq$ a known convergent	$\sum_{n=1}^{\infty} a_n \geq$ a known divergent	try to use p-series or geometric series to compare
Limit Comparison (LCT)	$\sum_{n=1}^{\infty} a_n$ $a_n > 0$	$\lim_{n\to\infty} \dfrac{a_n}{b_n} = L > 0$ & $\sum_{n=1}^{\infty} b_n$ is known to be convergent	$\lim_{n\to\infty} \dfrac{a_n}{b_n} = L > 0$ & $\sum_{n=1}^{\infty} b_n$ is known to be divergent	this version of LCT is inconclusive if $L = 0$ or $L = \infty$
Alternating (AST)	$\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ $b_n \geq 0$	(1.) $\lim_{n\to\infty} b_n = 0$ (2.) $b_{n+1} \leq b_n$	use TFD $\lim_{n\to\infty} (-1)^{n-1} b_n \neq 0$	$(-1)^{n-1}$ $= \cos\left((n-1)\pi\right)$
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert \dfrac{a_{n+1}}{a_n} \right\rvert = L < 1$	$\lim_{n\to\infty} \left\lvert \dfrac{a_{n+1}}{a_n} \right\rvert = L > 1$	inconclusive if $L = 1$ great for ! and $(\ )^n$
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = L < 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = L > 1$	inconclusive if $L = 1$ great for $(\ )^n$

If $\sum_{n=1}^{\infty} \lvert a_n\rvert$ converges, then $\sum_{n=1}^{\infty} a_n$ is **<u>absolute convergent</u>** (which implies $\sum_{n=1}^{\infty} a_n$ also converges)

If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} \lvert a_n\rvert$ diverges, then $\sum_{n=1}^{\infty} a_n$ is **<u>conditional convergent</u>**

2. Function and Continuity

2.1. The Notion of a Function

Def. function : a rule that assigns to every number x in a collection D, a number $f(x)$.

- The set D is called the domain of the function
- $f(x)$ is called the value of the function at x
- The set of all values of a function is called its range.
- The set of ordered pairs $(x, f(x))$ is called the graph of f

Def. A function f is bounded : $\exists m>0$ s.t. $|f(x)| \le m$.

A function g is bounded away from 0 : $\exists p>0$ s.t. $|g(x)| > p$

2.2. Continuity.

Def. A function is continuous at c

: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $|x-c|<\delta$, then $|f(x)-f(c)|<\varepsilon$

Def. The Limit of a function $f(x)$ as x tends to c is L

: $\lim_{x \to c} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0$ s.t. if $0<|x-c|<\delta$, then $|f(x)-L|<\varepsilon$.

Def. A function is continuous at c

$\Rightarrow$ We can also say $\lim_{x \to c} f(x) = f(c)$.

f is not continuous at c

: f is discontinuous at c.

2.2. Continuity. 계속

Thm 2.1. If $\lim_{x\to c} f(x) = L_1$, $\lim_{x\to c} g(x) = L_2$, then the following holds:

(a) $\lim_{x\to c} (f(x)+g(x)) = L_1 + L_2$.

(b) $\lim_{x\to c} f(x)g(x) = L_1 L_2$.

(c) If $L_1 \neq 0$, then $\lim_{x\to c} \frac{1}{f(x)} = \frac{1}{L_1}$

Thm 2.2. Squeeze Theorem.

If $f(x) \le g(x) \le h(x)$ for all x in open interval containing c, except possibly at $x = c$,

and if $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} g(x) = L$.

Thm 2.3. Suppose f, g, h is continuous at c, and $h(c) \neq 0$

Then $f+g$, fg, $\frac{f}{h}$ are continuous at c.

Def. A function f is continuous on an open interval (a,b)

: $\forall c \in (a,b)$ s.t. f is continuous at c.

A function f is continuous on an closed interval $[a,b]$

: f is continuous at an interval (a,b)

& $\forall \varepsilon > 0, \exists \delta > 0$ s.t. if $0 \le x - a < \delta$,

then $|f(x) - f(a)| < \varepsilon$ $(\Leftrightarrow \lim_{x\to a+} f(x) = f(a))$

& $\forall \varepsilon > 0, \exists \delta > 0$ s.t. if $-\delta < x - b \le 0$,

then $|f(x) - f(b)| < \varepsilon$ $(\Leftrightarrow \lim_{x\to b-} f(x) = f(b))$.

Def. A function f is called Uniformly continuous on an interval I : $\forall \varepsilon > 0, \exists \delta > 0$ s.t. if $|x - z| < \delta$, $x, z \in I$,

then $|f(x) - f(z)| < \varepsilon$.

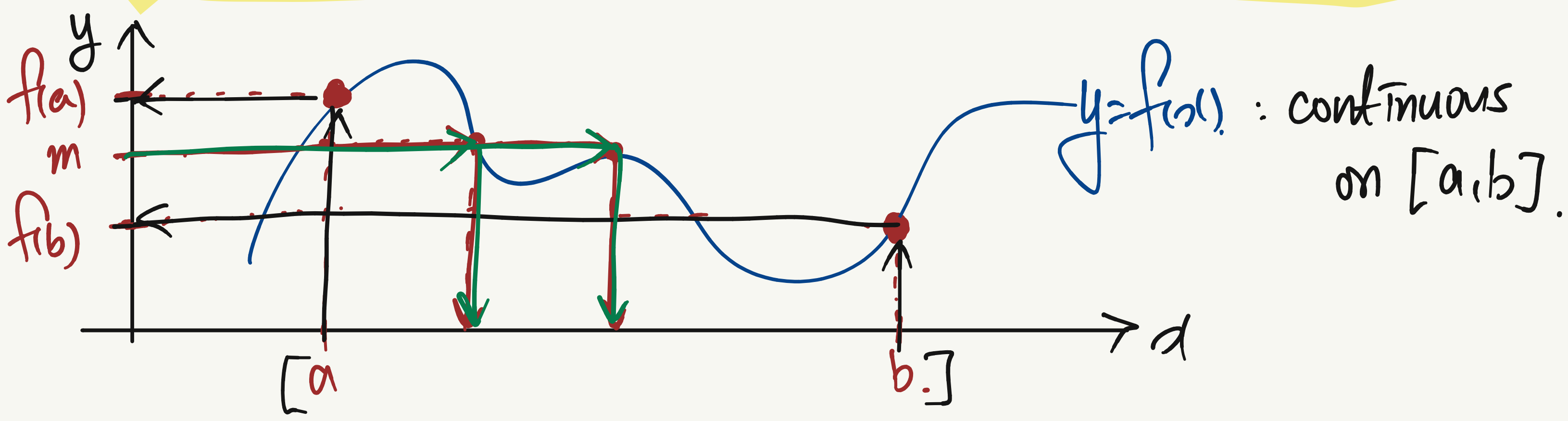
Thm. 2.4. If a function f is continuous on an closed interval $[a,b]$,

then f is uniformly continuous on $[a,b]$.

2.2. Continuity. 계속.

Thm. 2.5. The Intermediate Value Theorem (IVT) (사이값 정리)

If f is continuous on a closed interval $[a,b]$, then f takes on all values between $f(a)$ and $f(b)$.



Thm 2.6. The Extreme Value Theorem (EVT) (최대·최소 정리)

If f is continuous on a closed interval $[a,b]$, then f takes both a maximum value and a minimum value at some points in $[a,b]$.

Cor: If f is continuous on (a,b), and $\lim_{x\to a+} f(x) = \lim_{x\to b-} f(x) = \infty$, then f has minimum value at some point in (a,b).

Similarly, $\lim_{x\to a+} f(x) = \lim_{x\to b-} f(x) = -\infty$, then f has maximum value at some point in (a,b)

2.3. Composition and Inversions of Functions.

Def. Let f and g be two functions, and suppose that all range of g is included in the domain of f. Then

The composition of f with $g \Leftrightarrow f\circ g$ $:= (f\circ g)(x) = f(g(x))$.

Thm. 2.7. The composition of two continuous function is continuous.

Thm. 2.8. Suppose $f\circ g$ is defined on an open interval containing c, that $\lim_{x\to c} g(x) = L$, and that f is continuous at L.

Then, $\lim_{x\to c} (f\circ g)(x) = f(L)$, that is, $\lim_{x\to c} f(g(x)) = f(\lim_{x\to c} g(x))$

2.3. Composition and Inversions of Functions. 계속.

Def. A function g is called invertible

: If function g has the property that different inputs always lead to different outputs,
(i.e. $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$)
then we can determine its input from output.

Inverse function f of invertible function g
: the domain of f is the range of g.
$f(y)$ is defined as the number x for which $g(x) = y$.

- We denote f as g^{-1} (i.e. $f = g^{-1}$).

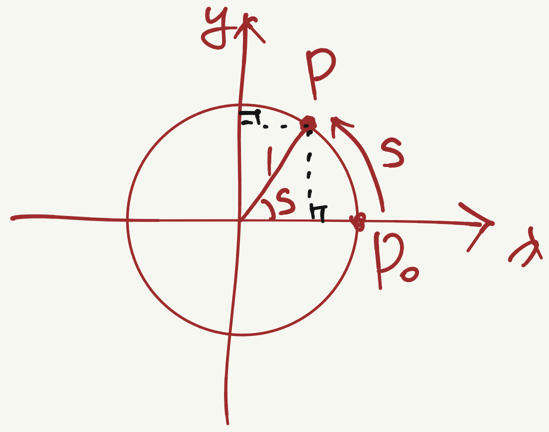
Def. A function f is increasing : $f(a) < f(b)$ whenever $a < b$.
A function f is decreasing : $f(a) > f(b)$ whenever $a < b$.
A function f is nondecreasing : $f(a) \leq f(b)$ whenever $a < b$.
A function f is nonincreasing : $f(a) \geq f(b)$ whenever $a < b$.
A function f is strictly monotonic : f is either increasing or decreasing.
A function f is monotonic : f is either nonincreasing or nondecreasing.

Lemma. If the function f is strictly monotonic, then f is invertible.

Thm. 2.9. Inversion Theorem.
Suppose that f is continuous and strictly monotonic function defined on an interval $[a, b]$.
Then, its inverse g is continuous strictly monotonic function defined on the closed interval between $f(a)$ and $f(b)$.

Def. Sine and Cosine.

Let's think Unit Circle which centered at origin.



s is the distance along the arc between P_0 and P.

Denote x and y-coordinates of $P(s)$ by $x(s)$ and $y(s)$.

We define $\begin{cases} \cos s = x(s) \\ \sin s = y(s) \end{cases}$.

Def. Exponential Function

: every continuous function f that satisfies

$f(x+y) = f(x)f(y)$ and $a = f(1) > 0$.

$\Rightarrow$ We can proof $f(x) = a^x$ for $x \in \mathbb{R}$.

↳ ① positive integer
② positive integer reciprocal.
③ positive rational number
④ Zero and negative rational number
$\Rightarrow$ rational number.
⑤ Thus f is continuous, irrational numbers
$\Rightarrow$ Real number

Thm. Exponential Growth. For $a > 1$,

$\lim_{x \to \infty} \frac{a^x}{x^k} = \infty$, no matter how k is big integer.

Def. Logarithm: Since $f(x) = a^x$ $(a \neq 1)$ is continuous strictly monotonic function, f is invertible.

we denote the inverse of $f(x) = a^x$ as $g(x) = \log_a x$.

When a is the number e, we defined before, Simply denote as $g(x) = \log x$.

2.6. Sequence of Functions and Their Limits

Def. A sequence of functions
: List of functions $f_1, f_2, f_3, \cdots$ with a common domain D

The sequence of functions f_n converges pointwise to a function f on D
: $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in D$

$\Leftrightarrow$ $\forall x \in D$, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. if $n > N$,
then $|f_n(x) - f(x)| < \varepsilon$.

The sequence of functions f_n converges uniformly to a function f on D
: $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. if $n > N$, then
$|f_n(x) - f(x)| < \varepsilon$ for all $x \in D$.

Thm. 2.11. Let $\{f_n\}$ be a sequence of functions, each continuous on the closed interval $[a,b]$.
If $\{f_n\}$ converges uniformly to f on $[a,b]$,
then f is continuous on $[a,b]$.

Thm. 2.12. Suppose $\{f_n\}, \{g_n\}$ are uniformly convergent sequences of continuous functions on $[a,b]$, converging to f and g.

Then, (a) $f_n + g_n$ converges uniformly to $f+g$ on $[a,b]$.

(b) $f_n g_n$ 〃 fg 〃

(c) If $f \neq 0$ on $[a,b]$, then ① for large n, $f_n \neq 0$.
② $\frac{1}{f_n}$ converges uniformly to $\frac{1}{f}$ on $[a,b]$.

(d) If h is a continuous function with range contained in $[a,b]$,
then $g_n \circ h$ converges uniformly to $g \circ h$ on $[a,b]$.

(e) If k is a continuous function on a closed interval contains the range of g_n and g, then $k \circ g_n$ converges uniformly to $k \circ g$ on $[a,b]$.

2.6. Sequences of Functions and Their Limits 계속.

Def. The sequence of functions $\{f_n\}$ exist, then we define

$\Rightarrow$ ① Partial Sums : $S_n = f_1 + f_2 + \cdots + f_n = \sum_{i=1}^{n} f_i$

② Series : $\lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} f_n$

if $\lim_{n\to\infty} S_n(x)$ exist, denote it by $f(x)$,

then we say the series converges to $f(x)$ at x.

We write $\sum_{n=0}^{\infty} f_n(x) = f(x)$.

If the sequence of partial sums converges uniformly on D,

then we say the series converges uniformly on D.

Def. A power series is a series of the form $\sum_{n=0}^{\infty} a_n (x-a)^n$

- Coefficient : The numbers a_n.
- Center of the power series : The number a.

Thm. 2.13. For a power series $\sum_{n=0}^{\infty} a_n (x-a)^n$, one of following must hold :

(a) The series converge absolutely for every x.

(b) $\quad$ 〃 $\quad$ only for $x=a$.

(c) There is a positive number R, called the radius of convergence, such that

the series $\begin{cases} \text{converges absolutely for } |x-a| < R \\ \text{diverges for } |x-a| > R. \end{cases}$

In case (c), the series might or might not converge at $x = a \pm R$

Thm 2.14. A power series $\sum_{n=0}^{\infty} a_n (x-a)^n$ converges uniformly to its limit function on every closed interval $[a-r, a+r]$, where $0 < r < R$, R is the radius of convergence.

In particular, the limit function is continuous on $(a-R, a+R)$

2.6. Sequences of Functions and Their Limits 계속.

Thm. (by Problem 2.67).

Consider a power series $\sum_{n=0}^{\infty} a_n x^n$.

Suppose $L = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$ exists and is positive.

Then, the radius of convergence (R) is $\frac{1}{L}$.

3. The Derivative and Differentiation.

3.1. The Concept of Derivative

Def. A function f is differentiable at a

: The limit $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists.

This limit is called derivative of f at a, and denoted by $f'(a)$

i.e. $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$.

Def. linear approximation to f at a

: $l(x) = f(a) + f'(a)(x-a)$

Thm. 3.1. A function is differentiable at a implies the function is continuous at a.

⊛ Some Uses for the Derivatives.

Speed	=	Rate of change of distance as a function of time.			
Velocity	=	〃	position	〃	time
Slope	=	〃	height	〃	horizontal distance
Current	=	〃	amount of electric charge		time

3.2. Differential Rules.

Thm.3.2. Derivative of sums and constant multiples.

If f and g are differentiable at x, and c is any constant,

then $f+g$, cf are differentiable at x, and

$(f+g)'(x) = f'(x) + g'(x)$. / $(cf)'(x) = cf'(x)$.

Thm.3.3. Product Rule.

If f and g are differentiable at x,

then their product is differentiable at x, and

$$(fg)'(x) = f(x)g'(x) + f'(x)g(x).$$

Thm 3.4. Power Rule

For every positive integer n,

$$(x^n)' = nx^{n-1}$$

Thm 3.5. Reciprocal Rule

If f is differentiable at x and $f(x) \neq 0$,

then $\frac{1}{f}$ is differentiable at x, and

$$\left(\frac{1}{f}\right)'(x) = \frac{-f'(x)}{\{f(x)\}^2}$$

Thm3.6. Quotient Rule.

If f and g are differentiable at x and $g(x) \neq 0$, then their Quotient is differentiable at x,

and $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{\{g(x)\}^2}$

3.2. Differential Rules. 계속.

Thm 3.7. Chain Rule.

If f is differentiable at $g(x)$ and g is differentiable at x, then $f\circ g$ is differentiable at x, and

$$(f\circ g)'(x) = f'(g(x))\cdot g'(x).$$

Thm. 3.8. Power Rule for Rational exponents.

For every rational number $r\neq 0$, and for every $x>0$, $(x^r)' = rx^{r-1}$

3.3. Derivative of e^x and $\log x$.

Thm. 3.9. $(e^x)' = e^x$.

Thm. $(\log x)' = \frac{1}{x}$ $(x>0)$ / $(\log|x|)' = \frac{1}{x}$ $(x\neq 0)$

Thm. 3.10. Power Rule (Complete!)

If $r\neq 0$ and $x>0$, then x^r is differentiable and $(x^r) = rx^{r-1}$.

Thm. 3.11. Suppose y is a function of x for which

$$y' = ky$$

where k is a constant. Then there is a constant C such that $y = Ce^{kx}$

(pf). we need to show that the function $\frac{y}{e^{kx}}$ is constant.

$$\Rightarrow \frac{d}{dx}\left(\frac{y}{e^{kx}}\right) = \frac{d}{dx}\left(ye^{-kx}\right) = y'e^{-kx} + y\cdot(-k)e^{-kx}$$

$$= (ky - ky)e^{-kx} = 0.$$

So, $\frac{d}{dx}\left(\frac{y}{e^{kx}}\right) = 0$, $y/e^{kx} = C$ constant. $y = Ce^{kx}$

3.4. Derivatives of Trigonometric Functions.

Thm. 3.12. $\sin' t = \cos t$ and $\cos' t = -\sin t$

Thm. 3.13. Denote by f a solution of $f'' + f = 0$ for which $f(c)$ and $f'(c)$ are both 0 at some point c. Then $f(t) = 0$ for all t.

(pf). $f'' + f = 0$. multiply both sides by $2f'$.

$\Rightarrow 2f''f' + 2ff' = 0$.

By
① $(f^2)' = 2ff'$
② $(f'^2)' = 2f''f'$,
$(f')^2 + f^2$ is constant.

$\Rightarrow$ at c, $0+0=0$ is constant.

so, $(f')^2 + f^2 = 0$ for every t.

so, $f' = f = 0$ for every t.

Thm. 3.14. Suppose f_1 and f_2 are two solutions of $f'' + f = 0$ and that there is a number c for which $f_1(c) = f_2(c)$ and $f_1'(c) = f_2'(c)$.

Then $f_1(t) = f_2(t)$ for every t.

(pf) let $f = f_1 - f_2$

then, $f(c) = 0$. $f'(c) = 0$. By Thm 3.13,

$f(t) = 0$ for every t. so, $f_1(t) = f_2(t)$ for every t.

Thm. 3.15. Addition law for the sine and cosine.

$\cos(t+s) = \cos t \cos s - \sin t \sin s$.

$\sin(t+s) = \sin t \cos s + \cos t \sin s$.

3.4. Derivatives of Trigonometric Functions.

Thm. Derivatives of Inverse Trigonometric Functions.

$$(\sin^{-1}x)' = \frac{1}{\sqrt{1-x^2}} \quad (-1<x<1)$$

$$(\cos^{-1}x)' = \frac{-1}{\sqrt{1-x^2}} \quad (-1<x<1)$$

$$(\tan^{-1}x)' = \frac{1}{1+x^2}$$

Thm. Derivatives of Various Trigonometric Functions.

$$\tan' x = \sec^2 x.$$

$$\sec' x = \sec x \tan x.$$

$$\cot' x = -\csc^2 x.$$

$$\csc' x = -\csc x \cot x.$$

Def. The hyperbolic cosine : $\cosh x = \dfrac{e^x + e^{-x}}{2}$

The hyperbolic sine : $\sinh x = \dfrac{e^x - e^{-x}}{2}$

Thm. $(\cosh x)' = \sinh x$. $(\sinh x)' = \cosh x$

Thm. 3.16. Suppose f_1 and f_2 are two solutions of $f'' - f = 0$ and that there is a number c for which $f_1(c) = f_2(c)$ and $f_1'(c) = f_2'(c)$.

Then $f_1(t) = f_2(t)$ for every t.

3.5. Derivatives of Power Series.

Thm. 3.17. Term-by-term differentiation.

If the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $-R < x < R$, then f is differentiable on $(-R, R)$, and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$